

BAKER DOMAINS FOR NEWTON'S METHOD

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ABSTRACT. We show that there exists an entire function without finite asymptotic values for which the associated Newton function tends to infinity in some invariant domain. The question whether such a function exists had been raised by Douady.

1. INTRODUCTION AND RESULT

Let f be an entire function. Newton's method for finding the zeros of f consists of iterating the function

$$N(z) := z - \frac{f(z)}{f'(z)}.$$

If ξ is a zero of f , then $N(\xi) = \xi$ and $|N'(\xi)| < 1$, so there is an N -invariant domain U containing ξ in which the iterates N^k of N converge to ξ as $k \rightarrow \infty$. (Here N -invariance of U means that $N(U) \subset U$.)

There may also be N -invariant domains in which the iterates of N tend to ∞ . A simple example is given by $f(z) = P(z) \exp Q(z)$ where P and Q are polynomials, with Q nonconstant. Then N is rational. Moreover, in the terminology of complex dynamics, the point at ∞ is a fixed point of multiplier 1 of N , and the iterates of N tend to ∞ in the Leau petals associated to this fixed point.

If f does not have the above form, then N is transcendental; see [2] for an introduction to the iteration theory of transcendental meromorphic functions. A maximal N -invariant domain where the iterates of N tend to ∞ is called an *invariant Baker domain*.

A simple example (cf. [3]) is given by functions f for which $f(z) \sim \exp(-z^n)$ as $z \rightarrow \infty$ in some sector $|\arg z| < \varepsilon$. Then $N(z) = z + (1/n + o(1))z^{1-n}$ and this implies that $N^k|_U \rightarrow \infty$ as $k \rightarrow \infty$ for some N -invariant domain U containing all sufficiently large real numbers. Note that $f(x) \rightarrow 0$ as $x \rightarrow \infty$, $x \in \mathbb{R}$. Thus 0 is an asymptotic value of f , the positive real axis being an asymptotic path. Figuratively speaking one might say that Newton's method believes that there is a zero of f at $+\infty$, and thus it yields a domain U containing all sufficiently large reals such that $N^k(z) \rightarrow +\infty$ for $z \in U$ as $k \rightarrow \infty$.

The question arises whether an entire function f must always have 0 as an asymptotic value if N has an invariant Baker domain. This question was raised by A. Douady and has been brought to our attention by J. Rückert. It has been shown by X. Buff and J. Rückert [4] that the answer to this question is positive in situations much more general than those given above. However, we shall show that this is not always the case.

1991 *Mathematics Subject Classification.* 30D05, 37F10, 65H05.

The first author was supported by the Alexander von Humboldt Foundation and by the G.I.F., the German-Israeli Foundation for Scientific Research and Development, Grant G -809-234.6/2003.

Theorem. *There exists an entire function f without finite asymptotic values such that $N(z) = z - f(z)/f'(z)$ has an invariant Baker domain.*

Moreover, f can be chosen to be of any preassigned order strictly between $\frac{1}{2}$ and 1.

We explain the basic idea of the construction. Using functions of the type introduced by S. K. Balašov [1], in §2 we construct an entire function f of order less than 1 (and in fact of any preassigned order strictly between $\frac{1}{2}$ and 1) which satisfies

$$f(z) \sim \sqrt[q]{z}$$

for some integer q and some branch of the q -th root as $z \rightarrow \infty$ in the spiraling region

$$S := \{re^{ic \log r + i\theta} : r > 1, |\theta| < \theta_0\},$$

where $c := \pi/\log(q-1)$ and $0 < \theta_0 < \pi$. Here the relation between c and q is such that S is invariant under $z \mapsto -pz$ where $p := q-1$. We shall see in §3 that

$$\frac{f'(z)}{f(z)} \sim \frac{1}{qz}$$

so that

$$N(z) = z - \frac{f(z)}{f'(z)} \sim -pz.$$

This will show that S contains an N -invariant domain in which the iterates of N tend to ∞ . Hence N has an invariant Baker domain. Finally we shall show in §4, using the Denjoy-Carleman-Ahlfors Theorem, that f has no finite asymptotic values.

2. THE CONSTRUCTION OF f

Let (a_k) be a sequence of complex numbers tending to infinity. For $r > 0$ let $n(r)$ be the number of a_k , taking account of repetition, in $|z| \leq r$. Let

$$\rho := \limsup_{r \rightarrow \infty} \frac{\log n(r)}{\log r}.$$

Equivalently, ρ is the exponent of convergence of the sequence (a_k) . It is well known that the canonical product Π whose zeros are the a_k has order ρ ; that is,

$$\rho = \limsup_{r \rightarrow \infty} \frac{\log \log M(r, \Pi)}{\log r}$$

where $M(r, \Pi) := \max_{|z|=r} |\Pi(z)|$ is the maximum modulus. There are standard results concerning the asymptotic behavior of Π if all a_k lie on one ray and if

$$(1) \quad n(r) \sim \Delta r^\rho$$

for some $\Delta > 0$ as $r \rightarrow \infty$. These results have been extended by S. K. Balašov [1] to the case where the a_k lie on a logarithmic spiral, say

$$(2) \quad a_k \in \{re^{ic \log r} : r \geq 1\},$$

where $c > 0$. We quote only a simplified version of Balašov's result [1, Theorem 1], as this suffices for our purposes.

Lemma 1. *Let (a_k) be a sequence satisfying (1) and (2). Suppose that ρ is not an integer. Let Π be the canonical product formed with the a_k . Then*

$$\lim_{r \rightarrow \infty} \frac{\log \Pi(re^{ic \log r + i\theta})}{r^\rho} = -\frac{2\pi i \Delta \exp(i\rho\theta/(1+ic))}{1 - \exp(i2\pi\rho/(1+ic))}$$

for $0 < \theta < 2\pi$ and a suitable branch of the logarithm, the convergence being uniform for $\varepsilon \leq \theta \leq 2\pi - \varepsilon$ if $\varepsilon > 0$. In particular,

$$(3) \quad \lim_{r \rightarrow \infty} \frac{\log |\Pi(re^{ic \log r + i\theta})|}{r^\rho} = h(\theta) := -2\pi \Delta \operatorname{Re} \left(\frac{i \exp(i\rho\theta/(1+ic))}{1 - \exp(i2\pi\rho/(1+ic))} \right).$$

Now let $\frac{1}{2} < \rho < 1$ and $\Delta > 0$. Choose $p \in \mathbb{N}$ such that

$$(4) \quad \mu := \frac{\rho}{1+c^2} := \frac{\rho}{1+(\pi/\log p)^2} > \frac{1}{2},$$

thus defining $c := \pi/\log p$. Note that since $\frac{1}{2} < \mu < \rho < 1$ we have $c < 1$ and hence $p > \exp(\pi) > 23$. Let (a_k) be a sequence satisfying (1) and (2) and let Π be the canonical product formed with the a_k so that (3) holds.

A series of elementary modifications of Π will produce the function f of our theorem.

A computation shows that

$$h(0) = -2\pi \Delta \operatorname{Re} \left(\frac{i}{1 - \exp(i2\pi\rho/(1+ic))} \right) = \frac{2\pi \Delta \exp(2\pi\mu c)}{|1 - \exp(i2\pi\rho/(1+ic))|^2} \sin(2\pi\mu).$$

Since $\frac{1}{2} < \mu < 1$ we thus have $h(0) < 0$. Hence there exists $\theta_0 > 0$ such that $h(\theta) < 0$ for $|\theta| < \theta_0$. For $0 < \varepsilon < \theta_1 < \theta_0$ we thus deduce from (3) that there exists $\eta_0 > 0$ such that

$$(5) \quad \log |\Pi(re^{ic \log r + i\theta})| \leq -\eta_0 r^\rho \quad \text{for } \varepsilon \leq |\theta| \leq \theta_1,$$

provided r is sufficiently large.

We show that an estimate of this type also holds for $|\theta| < \varepsilon$. In order to do so, we use a standard estimate which in slightly different form can be found in [6, p. 548] or [8, p. 117].

Lemma 2. *Let $D \subset \mathbb{C}$ be an unbounded domain. For $r > 0$ such that the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$ intersects D , let $r\theta(r)$ be the linear measure of that intersection. Let $\theta^*(r) := \theta(r)$ if $C_r \not\subset D$ and let $\theta^*(r) := \infty$ and thus $1/\theta^*(r) := 0$ if $C_r \subset D$.*

Suppose that $u : \overline{D} \rightarrow [-\infty, \infty)$ is continuous in \overline{D} and subharmonic in D . Suppose also that u is bounded above on ∂D , but not bounded above in D . Let $0 < \kappa < 1$ and let $R > 0$ be such that C_R intersects D . Then $B(r, u) := \max_{|z|=r} u(z)$ satisfies

$$\log B(r, u) \geq \pi \int_R^{\kappa r} \frac{dt}{t\theta^*(t)} - O(1)$$

as $r \rightarrow \infty$.

We may assume that ε in (5) is chosen such that $0 < \varepsilon < \pi/2$. We consider the spiralling domain

$$D := \{re^{ic \log r + i\theta} : r > 1, |\theta| < \varepsilon\}$$

and the function

$$u(z) := \log |\Pi(z)| + \eta_0 |z|^\rho.$$

Then u is continuous in \overline{D} , subharmonic in D and bounded above on ∂D . We claim that u is also bounded above in D . Otherwise, on applying Lemma 2 and noting that $\theta^*(r) = 2\varepsilon$ we find that

$$\log B(r, u) \geq \pi \int_R^{\kappa r} \frac{dt}{2\varepsilon t} - O(1) = \frac{\pi}{2\varepsilon} \log r - O(1) > \log r$$

and thus

$$\log M(r, \Pi) = B(r, u) - \eta_0 r^\rho > r - \eta_0 r^\rho > \frac{r}{2}$$

for large r . This implies that the order of Π is at least 1, a contradiction.

Thus u must be bounded above in D , and this, together with (5), implies that if $0 < \eta_1 < \eta_0$, then

$$(6) \quad \log |\Pi(re^{ic \log r + i\theta})| \leq -\eta_1 r^\rho \quad \text{for } |\theta| \leq \theta_1$$

and sufficiently large r .

Let L be the natural parametrization of the logarithmic spiral on which the a_k lie; that is, $L : [1, \infty) \rightarrow \mathbb{C}$, $L(t) = te^{ic \log t}$. Then $\Pi(L(t)) \rightarrow 0$ as $t \rightarrow \infty$ by (6). Thus there exists $t_0 > 1$ such that $|\Pi(L(t))| < |\Pi(L(t_0))|$ for $t > t_0$.

The function f of our theorem will now be defined as follows. We put $z_0 := L(t_0)$ and define $g_0(z) := \Pi(z + z_0)$. Next we put $q := p + 1$ and $g_1(z) := g_0(z^q)$, and define $\sigma : [0, \infty) \rightarrow \mathbb{C}$ by $\sigma(t) = \sqrt[q]{L(t_0 + t) - z_0}$, where the branch is chosen such that $\arg \sigma(t) = q^{-1} \arg(L(t_0 + t) - z_0)$ for $t > 0$. We then define

$$(7) \quad g_2(z) := \int_0^z g_1(\zeta)^n d\zeta = \int_0^z \Pi(\zeta^q + z_0)^n d\zeta,$$

where $n \in \mathbb{N}$. It will follow easily that

$$a := \int_\sigma g_1(z)^n dz$$

is finite for all $n \in \mathbb{N}$, and using a result of W. K. Hayman [5, Lemma 1] we will see that $a \neq 0$ if n is sufficiently large. For such a value of n we then define $g_3(z) := g_2(z)/az$ and note that g_3 is of the form $g_3(z) = g_4(z^q)$ for some entire function g_4 . The function claimed in the statement of the theorem is

$$f(z) := zg_4(z)^{q-1}.$$

To prove that f has the desired properties, we will determine the asymptotic behavior of the g_j and f in spiralling regions similar to D . We first note that it follows from (6) that if $0 < \eta_2 < \eta_1$ and if $0 < \theta_2 < \theta_1$, then

$$\log |g_0(re^{ic \log r + i\theta})| \leq -\eta_2 r^\rho \quad \text{for } |\theta| \leq \theta_2$$

and sufficiently large r . This implies that if $|\theta| \leq \theta_2/q$ and if r is sufficiently large, then

$$\log |g_1(re^{ic \log r + i\theta})| = \log |g_0(r^q e^{ic \log(r^q) + iq\theta})| \leq -\eta_2 r^{q\rho}.$$

With

$$S_1 := \left\{ re^{ic \log r + i\theta} : r > 1, |\theta| < \frac{\theta_2}{q} \right\}$$

we thus find that

$$|g_1(z)| \leq \exp(-\eta_2 |z|^{q\rho})$$

if $z \in S_1$ is sufficiently large. Moreover, we have $\sigma(t) \in S_1$ for large t , since

$$\begin{aligned} \arg \sigma(t) &= \frac{1}{q} \arg(L(t_0 + t) - z_0) \\ &= \frac{1}{q} \arg L(t_0 + t) + o(1) \\ &= \frac{c}{q} \log |L(t_0 + t)| + o(1) \\ &= c \log |\sigma(t)| + o(1). \end{aligned}$$

We deduce that the integral defining a converges for all $n \in \mathbb{N}$. In order to show that $a \neq 0$ for large n , we shall use the following result of W. K. Hayman [5, Lemma 1].

Lemma 3. *Let γ be a Jordan arc in \mathbb{C} which tends to ∞ in both directions and let g be holomorphic in a domain containing γ . Suppose that $\int_\gamma |g(z)| |dz| < \infty$ and that $|g(z)| \rightarrow 0$ as $z \rightarrow \infty$ on γ . Suppose also that $|g(z)| \leq M$ for z on γ , with equality for a single point z_1 on γ which satisfies $g'(z_1) = 0$. Suppose finally that γ cannot be deformed in a neighborhood of z_1 into a curve on which $|g(z)| < M$. Then*

$$\int_\gamma g(z)^n dz \neq 0$$

for all sufficiently large integers n .

We apply this lemma with $g := g_1$, the curve γ parametrized as $\gamma : \mathbb{R} \rightarrow \mathbb{C}$,

$$\gamma(t) := \begin{cases} \sigma(-t) & \text{if } t \leq 0, \\ \sigma^*(t) := e^{2\pi i/q} \sigma(t) & \text{if } t > 0, \end{cases}$$

and $z_1 := \gamma(0) = 0$. Since by the choice of z_0 we have

$$|g_1(\sigma^*(t))| = |g_1(\sigma(t))| = |\Pi(L(t_0 + t))| < |\Pi(L(t_0))| = |g_1(\sigma(0))|$$

for $t > 0$, it follows that $|g_1(\gamma(t))| < |g_1(z_1)|$ for $t \neq 0$. Moreover, $g'_1(z_1) = g'_1(0) = 0$, and thus the hypotheses of Lemma 3 are satisfied. Since

$$\int_\gamma g_1(z)^n dz = - \int_\sigma g_1(z)^n dz + \int_{\sigma^*} g_1(z)^n dz = (-1 + e^{2\pi i/q}) \int_\sigma g_1(z)^n dz$$

we conclude from Lemma 3 that $a = \int_\sigma g_1(z)^n dz \neq 0$ for sufficiently large values of n .

We deduce that $g_2(\sigma(t)) \rightarrow a$ as $t \rightarrow \infty$. More generally, $g_2(z) \rightarrow a$ as $z \rightarrow \infty$ in S_1 . In fact, if $z \in S_1$ then

$$g_2(z) - a = \int_{\tau_z} g_1(\zeta)^n d\zeta$$

for any path τ_z joining z to ∞ in S_1 . For large $z \in S_1$ and suitable τ_z we find that

$$|g_2(z) - a| \leq \int_{\tau_z} |g_1(\zeta)|^n |d\zeta| \leq \int_{\tau_z} \exp(-n\eta_2 |\zeta|^{q\rho}) |d\zeta| \leq \exp(-\eta_3 |z|^{q\rho})$$

for some $\eta_3 > 0$. It follows that if $z \in S_1$ is sufficiently large, then

$$\left| g_3(z) - \frac{1}{z} \right| = \frac{|g_2(z) - a|}{|az|} \leq \frac{\exp(-\eta_3 |z|^{q\rho})}{|az|} \leq \exp(-\eta_3 |z|^{q\rho}).$$

Now let

$$S_2 := \{ r e^{ic \log r + i\theta} : r > 1, |\theta| < \theta_2 \}.$$

For $z \in S_2$ we have $\sqrt[p]{z} \in S_1$ for a suitable branch. For large $z \in S_2$ we thus find that

$$\left| g_4(z) - \frac{1}{\sqrt[p]{z}} \right| = \left| g_3(\sqrt[p]{z}) - \frac{1}{\sqrt[p]{z}} \right| \leq \exp(-\eta_3|z|^\rho);$$

i. e. if $z \in S_2$ is sufficiently large, then

$$(8) \quad |f(z) - \sqrt[p]{z}| \leq \exp(-\eta_4|z|^\rho)$$

for some $\eta_4 > 0$ and a suitable branch.

Remark. We have chosen z_0 and n in the way described only to ensure that $a \neq 0$. In a generic situation we could probably define g_2 via (7) with $z_0 = 0$ and $n = 1$.

3. NEWTON'S METHOD FOR f

We choose θ_3 with $0 < \theta_3 < \theta_2$ and define

$$S_3 := \{re^{ic \log r + i\theta} : r > 1, |\theta| < \theta_3\}.$$

Then there exists $\delta > 0$ such that if $z \in S_3$ is sufficiently large, then the closed disk of radius $\delta|z|$ around z is contained in S_2 . With $d(z) := f(z) - \sqrt[p]{z}$ we deduce from (8) that if $z \in S_3$ is sufficiently large, then

$$\begin{aligned} \left| f'(z) - \frac{\sqrt[p]{z}}{qz} \right| &= |d'(z)| \\ &= \frac{1}{2\pi} \left| \int_{|\zeta-z|=\delta|z|} \frac{d(\zeta)}{(\zeta-z)^2} d\zeta \right| \\ &\leq \frac{1}{\delta|z|} \max_{|\zeta-z|=\delta|z|} |d(\zeta)| \\ &\leq \frac{1}{\delta|z|} \exp(-\eta_4(1-\delta)^\rho|z|^\rho) \\ &\leq \exp(-\eta_5|z|^\rho) \end{aligned}$$

for some $\eta_5 > 0$. Combining this with (8) we find that if $z \in S_3$ is sufficiently large, then

$$\left| \frac{f(z)}{f'(z)} - qz \right| \leq \exp(-\eta_6|z|^\rho)$$

where $\eta_6 > 0$. Recalling that $q = p + 1$ we deduce that

$$(9) \quad |N(z) + pz| = \left| z - \frac{f(z)}{f'(z)} + pz \right| = \left| \frac{f(z)}{f'(z)} - qz \right| \leq \exp(-\eta_6|z|^\rho)$$

for large $z \in S_3$. In particular,

$$||N(z)| - p|z|| \leq \exp(-\eta_6|z|^\rho)$$

which implies that

$$(10) \quad |\log |N(z)| - \log(p|z|)| \leq \exp(-\eta_6|z|^\rho)$$

for large $z \in S_3$. Moreover, (9) yields

$$(11) \quad |\arg N(z) - \arg(-pz)| \leq \exp(-\eta_6|z|^\rho)$$

for large $z \in S_3$. Recalling that c was chosen such that $c \log p = \pi$ we deduce from (10) and (11) that

$$\begin{aligned}
& \left| \arg N(z) - c \log |N(z)| \right| \\
& \leq \left| \arg N(z) - \arg(-pz) \right| + \left| \arg(-pz) - c \log(p|z|) \right| \\
& \quad + \left| c \log(p|z|) - c \log |N(z)| \right| \\
& \leq \left| \arg(-pz) - c \log(p|z|) \right| + (1+c) \exp(-\eta_6 |z|^\rho) \\
& = \left| \arg z + \pi - c \log p - c \log |z| \right| + (1+c) \exp(-\eta_6 |z|^\rho) \\
& = \left| \arg z - c \log |z| \right| + (1+c) \exp(-\eta_6 |z|^\rho) \\
& \leq \left| \arg z - c \log |z| \right| + \frac{1}{2|z|}
\end{aligned}$$

for large $z \in S_3$. Since $p > 23$ we deduce from (9) that $|N(z)| > 2|z|$ if $z \in S_3$ and if $|z|$ is sufficiently large, say $|z| > r_0 > 1$. Combining this with the previous estimate we conclude that if

$$\left| \arg z - c \log |z| \right| < \theta_3 - \frac{1}{|z|},$$

then

$$\left| \arg N(z) - c \log |N(z)| \right| < \theta_3 - \frac{1}{|z|} + \frac{1}{2|z|} = \theta_3 - \frac{1}{2|z|} < \theta_3 - \frac{1}{|N(z)|}$$

if $z \in S_3$ and if $|z|$ is large enough, say $|z| > r_1 > r_0$. This implies that

$$U := \left\{ r e^{i c \log r + i \theta} : r > r_1, |\theta| < \theta_3 - \frac{1}{r} \right\}$$

is N -invariant. Since $|N(z)| \geq 2|z|$ for $z \in U$, we have $|N^k(z)| \geq 2^k |z|$ for $z \in U$ and $k \in \mathbb{N}$. Thus $N^k|_U \rightarrow \infty$ as $k \rightarrow \infty$. Hence U is contained in an invariant Baker domain of N .

4. ASYMPTOTIC VALUES OF f

Suppose that f has a finite asymptotic value, say $f(z) \rightarrow b \in \mathbb{C}$ as $z \rightarrow \infty$ along a curve Γ . The function

$$F(z) := \frac{f(z)^q}{z}$$

is entire since $f(0) = 0$. By (8) we have $F(z) \rightarrow 1$ as $z \rightarrow \infty$ along the logarithmic spiral L while $F(z) \rightarrow 0$ as $z \rightarrow \infty$ along Γ . Thus F has two finite asymptotic values. By the Denjoy-Carleman-Ahlfors Theorem (see [7, §XI.4.5]), F has order at least 1. On the other hand, F has the same order as f , which has been taken less than 1. This is a contradiction.

Remark. Our method will produce examples f of any preassigned non-integer order $\rho > 1$, as well as examples with more than one invariant Baker domain. We only sketch the modifications that have to be made.

We again choose ρ and p such that (4) is satisfied. The condition $\mu < 1$ need not be satisfied, and there may be several, say ℓ , intervals where $h(\theta)$ is negative and corresponding spiraling regions S_1, \dots, S_ℓ where $\Pi(z) \rightarrow 0$ as $z \rightarrow \infty$. It is not difficult to see that ℓ can be any given positive number. Let L_j be a curve starting at 0 which outside the unit circle is a logarithmic spiral in S_j and which inside the unit circle is a straight line from 0 to the corresponding point of the unit

circle. Deforming one of the curves L_j if necessary we may assume that there exists $z_0 \in \bigcup_{j=1}^{\ell} L_j$ such that $|\Pi(z_0)| > |\Pi(z)|$ for all $z \in \bigcup_{j=1}^{\ell} L_j$. Defining g_2 by (7) for some large n and defining f then as in §2 we find that $f(z) \sim c_j \sqrt[n]{z}$ for some $c_j \neq 0$ as $z \rightarrow \infty$ in S_j . As before we deduce that $N(z) \sim -pz$ as $z \rightarrow \infty$ in S_j , $j = 1, \dots, \ell$. We thus obtain an entire function f for which N has ℓ invariant Baker domains. A difference occurs in the proof that f does not have finite asymptotic values. Here we cannot simply appeal to the classical Denjoy-Carleman-Ahlfors Theorem, but instead use that the function f constructed has only ℓ “tracts”; see [6].

Balašov’s result takes a different form if ρ is an integer, but it seems possible to treat this case along the same lines.

Acknowledgments. We thank Johannes Rückert for bringing Douady’s question to our attention, and for interesting discussions about it. We also acknowledge helpful conversations with Alexandre Eremenko.

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